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Universal diamagnetism : a diagnostic test for spinless bosonic excitations

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Abstract : It is well known that a classical gas of point charged particles do not show any effect of magnetic field in thermal equilibrium. Magnetism is truly a quantum phenomenon. We demonstrate explicitly that diamagnetism is a robust property of charged spinless bose systems. It will be shown that diamagnetism is a universal phenomenon of spinless bose systems at finite temperature regardless of their interactions. Starting from a single particle, the discussion will go on to N particles system and finally, end on to an interacting field theory in an arbitrary spatial dimension d. We will present an exact and non-perturbative result of the theory which is an interesting feature from the field theory aspect. In this theory, as a byproduct, we have also been able to develop a novel regularization scheme, particularly useful for magnetic field problems. Finally, as an application we will take up the cases of elementary bosonic type of excitations in condensed matter physics-one is the cooper pairs in superconductivity and other is the composite bosons in Fractional Quantum Hall Effect.

Keywords : Diamagnetism, spinless bosons, cooper pairs, composite bosons.

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1. Introduction

To understand a quantum many body system, elementary excitations [1–3] play an important role. It was Debye who first introduced the concept of elementary excitations in lattice vibration so called the phonon theory. The elementary excitations have two essential characteristics as pointed out by Anderson [4]. In quantum many body theory, one is interested on the relative position of lowest excited states compared to the ground state. These lower states are important because they can be easily excited at very low temperature. It happens so that the thermodynamic and transport behaviour at low temperature are determined solely by these lowest excited states. Secondly, these states in a large system behave independently and hence can be treated as noninteracting ones. This fact simply helps one to understand the behaviour of the complicated many bose systems in terms of these excitations.

Elementary excitations can be classified as fermionic and bosonic nature. The simplest example of fermionic excitations are excitations in metals near the Fermi surface. In case of strongly correlated electron systems, it is possible to think in terms of elementary excitations very close to the Fermi surface. These quantum excitations termed as Landau Quasi particles interact very weakly through an interaction $f(k,k')$ In this theory, one can express quite a number of physically observable quantities such as specific heat, magnetic susceptibility and transport properties. The effective mass of the quasi particle is modified by this interaction.

The second category of elementary excitations originates from electrons are the collective one which are

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bosonic in nature. It is amazing to note that though the elementary particles in the system are fermions in nature but the collective excitations could be bosonic. The simplest example is the quantum of lattice vibrations-phonons.

In this paper, we want to investigate the nature of these elementary excitations by noting their behaviour under an external magnetic field. If the excitations show diamagnetic nature then we can infer that the nature of the excitation is bosonic. We also give two simple examples from condensed matter systems at low temperature.

2. Classical case

The response of a collection of classical point charged particle in presence of an external magnetic field in thermal equilibrium is non-magnetic [5]. This theorem is known as Bohr-van Leewen theorem. Later, Peier's [6] gave a very simple elegant argument based on partition function to conclude that the partition function is unaffected by the application of an external magnetic field. A pictorial proof [6] in terms of closed orbits of the inner electrons and the orbits at the boundary of the systems also shows the cancellation of two types of currents resulting zero magnetisation. There also exists another simple connection [7] between Brownian motion and magnetism where magnetic field is used as a some kind of counter in measuring the typical closed area in a specific Brownian motion. In this description, a typical high temperature limit gives the Bohr-van-Leewen theorem. Below, we present a simple proof of classical interacting N particle system in terms of partition function.

The N particle Hamiltonian is given by

$$H_N = \sum_{i=1}^{N} \frac{1}{2m_i}\left(p_i - \frac{e_i A(r_i)}{c}\right. \quad \cdot \sum_{i=1}^{N} V(r_i)$$

$$\sum_{i<j} W\left(\left|r_i - r_j\right|\right). \tag{1}$$

Here, A is the vector potential $V(r_i)$ is the one body potential and W $(|r_i - r_j|)$ is the two body pairwise interaction. The partition function of this system can be written as

$$Z_N(A) = \frac{1}{N!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{l=1}^{N} d^3p_i d^3r_i \exp(-\beta H_N). \tag{2}$$

With change of momentum variables, it is easy to notice that

$$Z_N(A) = \frac{1}{N!}\int\int\prod_{l=1}^{\cdot\cdot} m_i^3 d^3 v_i d^3 r_i$$

$$\times \exp\left| -\beta\right| \sum \frac{1}{2} m_i v_i^2 = V(r_i)$$

$$\times \exp\left| -\beta \sum_{i<j} W\left(\left|r_i - r_j\right|\right)\right)$$

$$Z_N(0). \tag{3}$$

Thus, the free energy is unaffected by the application of an external magnetic field. This points out that the effect of magnetic field in thermal equilibrium must be sought in quantum mechanics.

3. Single particle case

The physics of a system in presence of an external magnetic field is interesting from the theoretical point of view because of the invalidity of perturbation technique [8]. Before we go on to N particle systems, we briefly summarise the single particle picture. In this single particle picture, statistics do not come into play. But the effect of magnetic field on the systems is two fold–namely the orbital motion of the particle and the spin part. We know that diamagnetic effect comes solely from the orbital part while paramagnetism from the spin part. A simple statistical mechanical calculation shows that the free energy for the orbital part is given by

$$F_{\text{orbit}}(B,\beta) = F_{\text{orbit}}(0) + \beta^{-1}\log\left|\frac{\sinh\left(\beta\hbar\omega\right.}{\frac{\beta\hbar\omega}{2}}\right. \tag{4}$$

while for that of spin part is simply

$$F_{\text{spin}}(B,\beta) = -\beta^{-1}\log(2\cosh(\beta\hbar\omega/2)), \tag{5}$$

where $\omega = \frac{eB}{mc}$. This implies immediately that

$$\chi_{\text{orbit}}(B,\beta) < 0, \quad \chi_{\text{spin}} B, \beta > 0. \tag{6}$$

If one looks carefully to the Hamiltonian of single particle in an external uniform magnetic field, one finds that there are two competing terms given by

$$H = \frac{p^2}{2m} - \frac{eB\hbar L_z}{2mc} + \frac{e^2B^2}{8mc^2}\left(x^2+y^2\right). \tag{7}$$

A quick look at the above equation reveals that the second term is responsible for paramagnetism while the third term is for diamagnetism. Note carefully the change of sign before the two terms. In most atomic situations the term B^2 is unimportant because of the fact that only at fields of 10^5 Tesla or higher (this field is much larger than the fields produced in the laboratory), the two terms are comparable. The ground state of many systems has no angular momentum and for them the first order Zeeman effect vanishes exactly. But the first order correction to orbital part $<x^2 + y^2>$ is always non-zero and positive and almost same order of magnitude for all atomic systems. However, for non-zero angular momentum of the ground state of many particle system, it is not a priori clear which term dominates over the other to decide the response of the systems to an external uniform magnetic field. However, for any quantum systems, inspite of its very low magnitude, the existence of its diamagnetism is definite resulting a universal phenomena.

4. N (finite) particle system

Simon [9] in 1876 first demonstrated through Kato's inequality that N spinless bosons in an external magnetic field show diamagnetism. For the sake of completeness, we reproduce briefly Simon's arguments below for the ground state of the quantum system. The starting Hamiltonian is

$$H_N = -\sum_{i=1}^{N}\frac{1}{2m_i}\left(\nabla_i - \frac{e_iA(r_i)}{c}\right)^2 + \sum_{i=1}^{N}V(r_i)$$

$$+\sum_{i<j}W\left(\left|r_i - r_j\right|\right). \tag{8}$$

Let $\Psi(r_1, r_2, \ldots\ldots r_N)$ be the wave function. Since $|\Psi|^2 = (\Psi^*\Psi)$, we note that using 3N dimensional gradients

$$\begin{aligned}(|\Psi|\nabla(|\Psi|) &= |\mathrm{Re}(\Psi^*\nabla\Psi)| \\ &= |\mathrm{Re}[\Psi^*(\nabla - ieA/c)\Psi]| \\ &\le |\Psi||(\nabla - ieA/c)\Psi|. \end{aligned} \tag{9}$$

Therefore, with x denoting the N vectors $(r_1, r_2, \ldots\ldots r_N)$, we get

$$\nabla\ |\Psi|^2(x) \le |\ (\nabla - ieA/c)\Psi|^2(x). \tag{10}$$

It follows automatically then that

$$\int dx\,[(\nabla\ |\Psi|^2 + V(x)|\Psi|^2 + W(x)|\Psi|^2] \le$$

$$\int dx\,[|(\nabla - ieA/c)\Psi|^2 + V(x)|\psi|^2 + W(x)|\Psi|^2]. \tag{11}$$

Now, if one chooses Ψ to be the ground state wave function for H_N, then the right hand side equal to $E(A)$, whereas by the variational principle for the ground state energies, the left hand side is greater than $E(0)$. Hence, we get

$$E(0) \le E(A). \tag{12}$$

It may be noted that the above proof fails for fermions. The replacement of Ψ by $|\Psi|$ used above is not allowed by Fermi statistics. Even spinless fermions do not obey this inequality. For a counter example, we choose a spherically symmetric potential V and concentrate on a particular eigenstate $n = 1$. Then, energies of $l \neq 0$ states decrease in lowest order perturbation theory for a suitable choice of $\boldsymbol{A}$. This shows that the above inequality fails for fermions. The above proof does not depend on the nature of applied external magnetic field and is also independent of the nature of interaction between the particles. Later on, this proof was again confirmed *via* the connection between Brownian motion and magnetism [7]. Diamagnetism was also shown to be an integral property of hard core bosons[10]. It was shown in all the above proofs of the spinless bosons systems that the free energy of N particles in an external magnetic field is greater than the free energy without an external magnetic field. Thus, the free energy of such a system always increases with the magnetic field resulting the negative susceptibility. However, it is not clear whether the free energy is a *monotonically* increasing function of the magnetic field B or not.

All the above systems deal with N finite particle resulting the energy to be compared as finite. However, in a continuum field theory the degrees of freedom being infinite, one has to be very careful about the comparison of the two energies. The energies are divergent in nature (in presence or absence of magnetic field) and one has to develop a suitable regularization scheme in such a way that the difference of them is finite and cut-off independent. A simple example [11] from mathematical physics illustrates this idea used earlier in Casimir's effect. We have developed a novel regularisation scheme[12–14] particularly suitable for dealing this magnetic field problem.

5. Many particle systems-bosonic field theroy

There are several reasons for studying the scalar fields in an external magnetic field at any finite temperature. First of all the results of N particle system do not always translate in same form to a corresponding field theory [14]. Secondly, in field theory one has to develop a proper regularisation scheme to have a finite value of the observable (in this case the energy). Thirdly, to study the elementary excitations in the system in the long wavelength limit, it is better to consider the corresponding field theoretic version rather than the particle picture. More ambitiously, to look into the various symmetries being broken, it is easier to understand from the field theoretic version rather than the N particle picture.

The relativistic action of complex charged scalar field in the imaginary time formalism can be written as

$$S = \int\int d^d x d\tau \left[\left(D_\mu \Phi\right)\left(D^\mu \Phi\right)^* + m^2\left(\Phi\Phi^*\right) + V^2\left(\Phi^*\Phi\right)\right]. \tag{13}$$

Here μ = 0, 1, 2 and $D_\mu = \partial_\mu - ieA_\mu$. The imaginary time τ has the range from 0 to $\beta = \frac{1}{k_B T}$. The vector potential is A and the respective magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. The bosonic condition is imposed *via* the boundary condition $\Phi(x,0) = \Phi(x,\beta)$. Without the interaction, the partition function defined as

$$Z(A) = \int\int D[\Phi] D[\Phi^*] \exp(-S) \tag{14}$$

can be evaluated exactly. With proper regularization scheme, we have shown explicitly [12] that in two dimensions the free energy of this charged scalar fields in presence of a uniform magnetic field is greater than that without the magnetic field. This establishes the diamagnetic character of the charged scalar fields. In an arbitrary dimension d, for an interacting case, we have followed a non-perturbative approach [12,15] to write the partition function as

$$\frac{Z(A)}{Z(0)} = \langle\langle \exp(-S_{\rm int}) \rangle\rangle \leq \tag{15}$$

where the action S has been divided into parts S_0 which does not contain A and $S_{\rm int}$ containing A only. From the above eq. (15), it is easy to notice that $F(B) \geq F(0)$

6. Diagnostic test of two systems

We here discuss the two systems where elementary excitations are charged and bosonic in nature. One is the cooper pair in superconductivity and another is the composite bosons in Fractional Quantum Hall effect [16,17].

Cooper paris in superconductivity :

We have used the concept of universal diamagnetism to compute the diamagnetic susceptibility of anisotropic [15] as well as isotropic [12] superconductor above T_c. The low temperature superconductors are in general isotropic in nature while the high temperature superconductors are highly anisotropic. The relevant Landau-Ginzburgh free energy functional is given by

$$f(\phi,\alpha_1,\alpha_2,B,T) = \int d^d x\left[\frac{1}{2m}\left|\left(-i\hbar\nabla - ieA\right)\phi\right|^2 + \frac{1}{2}\left(\alpha_1^2 + \alpha_2^2\right)\left|\phi\right|^2\right], \tag{16}$$

where $\alpha_1^2 = \alpha_0 (T - T_c)^{\nu_1}$ and $\alpha_1^2 = \alpha_0 (T - T_c)^{\nu_2}$ and α_0 being a constant independent of temperature. The expression of the susceptibility obtained in this anisotropic case [15] as

$$\chi_d^{\rm aniso} \quad \frac{(d-1)^{(4-d)/2}(T-T_c)^{-(\nu_1+\nu_2)(4-d)/2}}{\left[A(d)(T-T_c)^{-\nu_1} B(d)(T-T_c)^{-\nu_2}\right]^{4-d)/2}}. \tag{17}$$

Here, $A(d) = d - 1 - \theta(d - 2)$ and $B(d) = \theta(d - 2)$. We have assumed that in plane correlation length and out of plane correlation length diverge at the same critical temperature with exponents ν_1 and ν_2 respectively. The isotropic case follows from the above expression when $\nu_1 = \nu_2 = \nu$. Note the symmetric nature of the expression of the susceptibility with interchange of ν_1 to ν_2. It will be interesting to study the fluctuation conductivity of superconductors in this model.

Composite bosons in FQHE :

The composite bosons [18] are the another interesting charged elementary excitations in 2d electron system in a strong magnetic field at low temperature. Electrons moving in two dimensions in such a strong magnetic field are mathematically equivalent to a collection of composite bosons in a much weaker magnetic field. In two dimensions a fermion can be treated as a boson to which an odd number of flux quantum $\left(\phi_0 = \frac{hc}{e}\right)$ is attached. This magnetic field is introduced *via* the statistical gauge field. This is statistical because of the fact that the statistical nature of the particles is changed on introduction of this gauge field. The binding of the statistical flux to the boson is achieved by the Chern-Simons interaction. Thus, apart from the external magnetic field in this system, we have another magnetic field originated by the statistical vector potential $\boldsymbol{a}$. This statistical magnetic field is proportional to the density of electrons and is generated internally. Mathematically, the statistical magnetic field $\boldsymbol{b}$ can be written as

$$\boldsymbol{b} = \nabla \times \boldsymbol{a} = k\rho\phi_0, \tag{18}$$

where the density $\rho(r) = \sum_i \delta^2(r_i - r)$ and $k = 2n + 1$, $n = 0, 1, 2$......... These Chern-Simon bosons do obey the usual diamagnetic inequality [15]. The advantage of mapping is clear if one considers the magic Landau level filing fractions where the fractional quantum Hall effects are observed. At all these filling factors, the Chern-Simons gauge field cancels on an average the external applied magnetic field B. Thus, the composite bosons in this situation experience a net zero magnetic field. As one changes the magnetic field from the value corresponding to magic filling factors (where composite bosons condensate), the condensed state try to repel the magnetic field so that the condensations remain. This explains the plateaus naively.

Let us examine the validity of mean field theory. The statistical interaction is of long range nature. To deal this long range interaction, it is sometimes useful to replace the effect of many distant particles by some sort of mean field. This automatically implies that we are replacing the many singular flux tubes by a smooth magnetic field with the same flux density. Hence, the average magnetic field is given by

$$b \approx \phi_0\rho. \tag{19}$$

In this description, the fluctuation of density of electrons is related to the statistical gauge field fluctuations. In such a magnetic field, the cyclotron radius is simple $r = \frac{mv}{eb}$ and the 2d Fermi velocity is given by $v_F = \frac{\hbar\sqrt{4\pi\rho}}{m}$. Now for a typical 2d density of electrons $\rho \sim 10^{11}$ cm^{-2}, we get $b \sim 10^4$ Gauss and $\rho \sim 10^{-14}$ cm and the magnetic length $l = \sqrt{\frac{\hbar c}{eb}} \sim 10^{-5}$ cm. We notice that $l >> r$.

Hence, the mean field theory is justified.

In certain Bi HTSC ceramics [19], a paramagnetic Meissner effect [20] (PME) is found in the field-cooling mode below a field of order 1 Oe. PME has also been observed in Nb, YBCO and LCO. Contrary to ordinary Meissner effect, they seem to turn on below T_c without the need of an external magnetic field. This PME is not caused by paramagnetic or ferromagnetic impurities. This is because of the fact that no large PME magnetisation is seen to start abruptly at T_c and no Curie-Weiss type susceptibility is seen above T_c. As has been pointed out earlier that the free energy could in principle decreases within a very narrow window of magnetic field retaining the higher value in compared to at the origin, one might sort in ϕ^4 field theory perturbatively to compute the partition function. This may shed some light on the typical elementary excitations in these materials.

To summarise, we have shown here that if the elementary excitations of a system are bosonic in nature then their response towards an external magnetic field is diamagnetic. We have given here two examples from strongly correlated electron systems to justify the above statement.

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References

[1] L D Landau and E M Lifshtiz *Statistical Physics* (New Youk : Pergamon Part II (1958)

[2] P Nozieres *Theory of Interacting Fermi Systems* (New York : Addison-Wesley) (1964)

[3] D Pines and P Nozieres *The Theory of Quantum Liquids* (Benjamin) (1966)

[4] P W Anderson *Concepts in Solids* (Singapore : World Scientific) (1997)

[5] J H van Leuwen *J. Phys.* (Paris) **2** 361 (1921)

[6] R E Peierls *Surprises in Theoretical Physics* (Princeton : Princeton University Press) Part I (1992)

[7] S Sinha and J Samuel *Phys. Rev.* **B50** 13871 (1994)

[8] D Jana *Applicability of Perturbation Theory to Many Body Physics, Physics Teacher* (2003) (accepted)

[9] B Simon *Phys. Rev. Lett.* **36** 1083 (1976); B Simon *Indiana Univ. J. Math. Phys.* **26** 1067 (1977)

[10] B S Shastry *Mod. Phys. Lett.* **B6** 1427 (1992)

[11] D Jana *Phys. Education* **18** 177 (2001)

[12] D Jana *Nucl. Phys.* **B473** [FS] 659 (1996)

[13] D Jana *Nucl. Phys.* **B485** [FS] 747 (1997)

[14] D Jana. *Fizika* (Zagreb) **B11** 63 (2002)

[15] D Jana *Int. J. Mod. Phys.* **B15** 2811 (2001)

[16] R E Prange and S M Girvin *The Quantum Hall Effect* (Berlin : Springer) (1987)

[17] T Chakraborty and P Pietiläinen *The Fractional Quantum Hall Effect* (Berlin : Springer-Verlag) (1998); For a modern view points, see S M Girvin *The Quantum Hall Effect : Novel Excitations and Broken Symmetries*, cond-mat/9907002

[18] S C Zhang, T H Hansson and S A Kivelson *Phys. Rev. Lett.* **62** 82 (1989); S C Zhang *Int. J. Mod. Phys.* **B6** 25 (1992); S. Kivelson, D H Lee and S C Zhang *Sc. Am.* **274** 64 (1996)

[19] W Braunisch, N Knauf, V Kataev, S Neuhausen, A Grutz, A Kock, B Roden, D Khomskii and D Wohlleben *Phys. Rev. Lett.* **68** 1908 (1992)

[20] M Sigrist and T M Rice *Rev. Mod. Phys.* **67** 503 (1995)